Gompertz Maximum Likelihood Estimation of Truncated Death Distribuitons

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Big Picture

Our challenge

We see only part of the picture (e.g., deaths aged 70 to 87).

No estimates of who died before or after

- How can we estimate death rates without denominators?
- How can we estimate $e(65)$ differences between groups?

Our idea

We can combine

- observed distribution of deaths (over limited range)
- our external knowledge of human mortality age-patterns

The hope is that this will produce good estimates of mortality rates, of $e(65)$, and of differences between groups

Today's agenda

- Intro to Maximum Likelihood Estimation
- Example with simulated data
- Attempt at validation with HMD
- Preliminary try at NUMIDENT
- Lessions and directions

Truncated Maximum Likelihood (in theory)

Philosophy

For given data X, we can a likelihood associated with a particular value of parameter θ.

We then choose the $\hat{\theta}$ to maximize this likelihood.

A simple example

Likelihood for observation i with value x_i:

$$L_i = L(\lambda|x_i) = f_\lambda(x_i)$$

Likelihood for all observations:

$$L = \prod_i L_i$$

Log-likelihood:

$$\mathcal{L} = \sum_i \log L_i$$

If we observe $x_1 = 3$ and $x_2 = 5$, then

$$L(\lambda|x_1) = \lambda e^{-3\lambda}$$
$$L(\lambda|x_2|) = \lambda e^{-5\lambda}$$
$$L(\lambda|x_1, x_2) = \lambda^2 e^{(} - 8\lambda)$$

$$\mathcal{L} = \sum_i \log L_i = 2 \log \lambda - 8\lambda$$

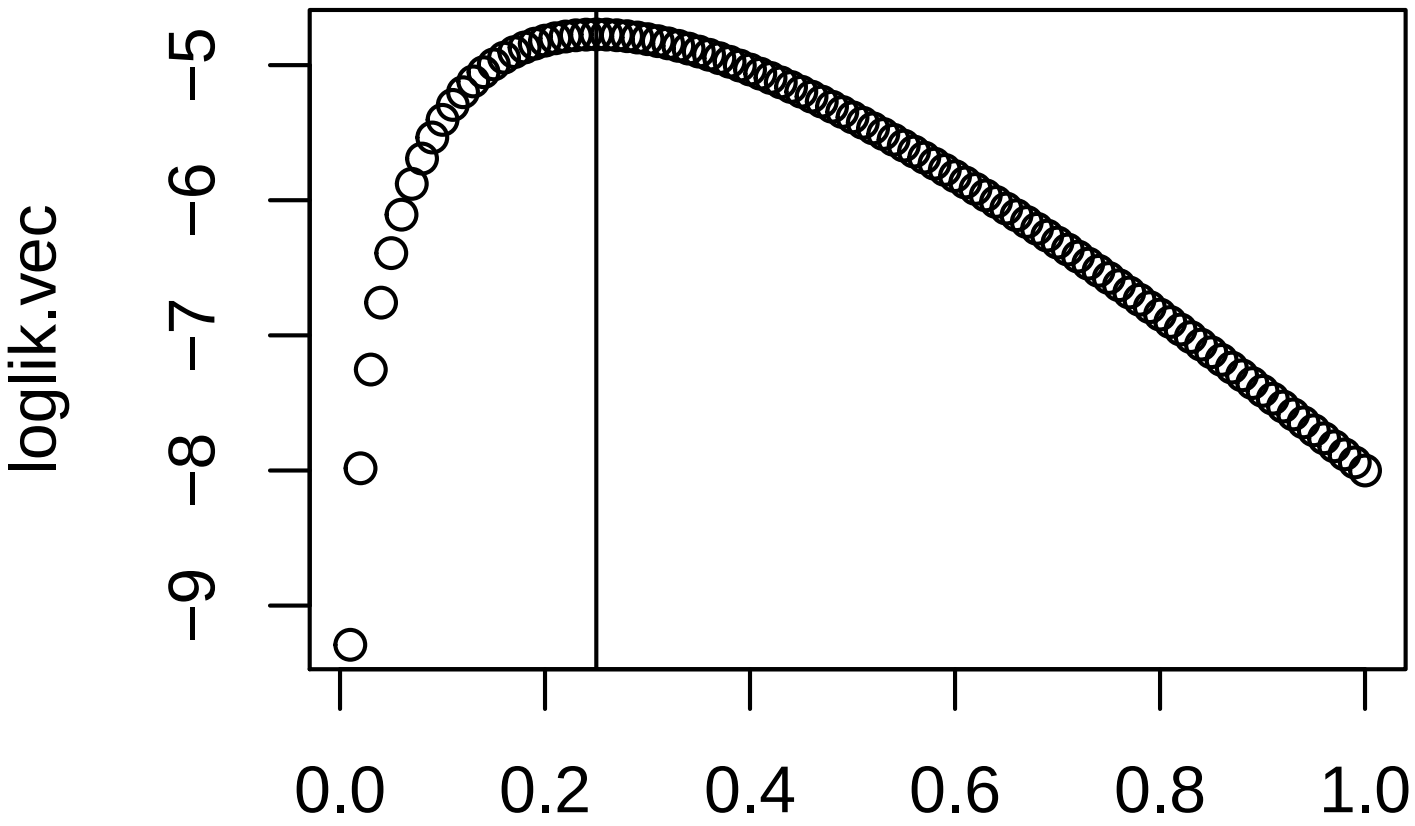
$$\frac{d\mathcal{L}}{d\lambda} = \frac{2}{\lambda} - 8 = 0$$

So,

$$\hat{\lambda}_{MLE} = 2/8 = 0.25$$

We did this by hand, but can also do with the computer

```
lambda.vec <- seq(.01, 1, .01)
loglik.vec = 2 * log(lambda.vec) - 8 * lambda.vec
plot(lambda.vec, loglik.vec)
abline(v = lambda.vec[which.max(loglik.vec)])
```



For truncated distribution we observe only from a to b

We can define the conditional distribution

$$f_{trunc} = \frac{f_\theta(x)}{\int_a^b f_\theta(x)\,dx} = \frac{f_\theta(x)}{F_\theta(b) - F_\theta(a)}$$

with likelihood

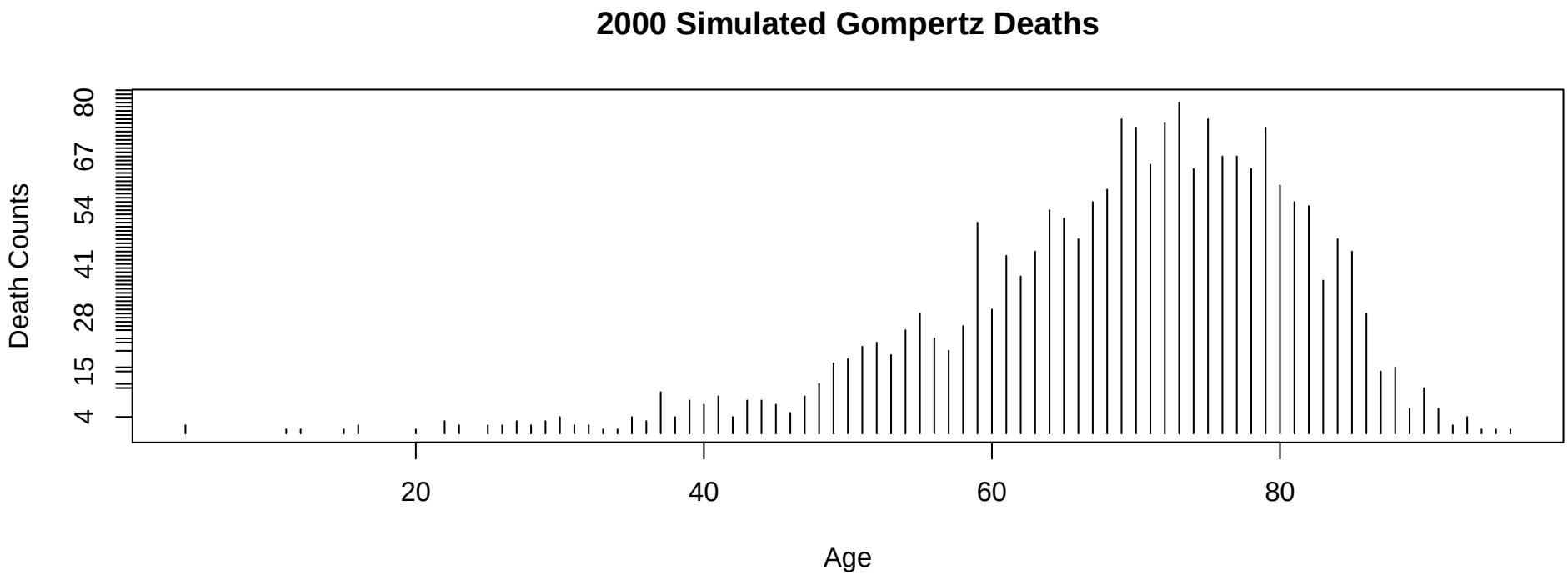
$$L(\theta|\mathbf{x}) = \prod \frac{f_\theta(x_i)}{F_\theta(b) - F_\theta(a)}$$

And then we maximize that.

Validation with Simulated Gompertz

Simulated Gompertz, without truncation

```
source("hmd_validation_functions.R")
N = 2000
set.seed(13)
x <- rgompertz.M(N, b = 1/10, M = 75)
Dx <- table(floor(x))
plot(names(Dx), Dx, type = "h",
     ylab = "Death Counts", xlab = "Age",
     main = "2000 Simulated Gompertz Deaths")
```



MLE estimation

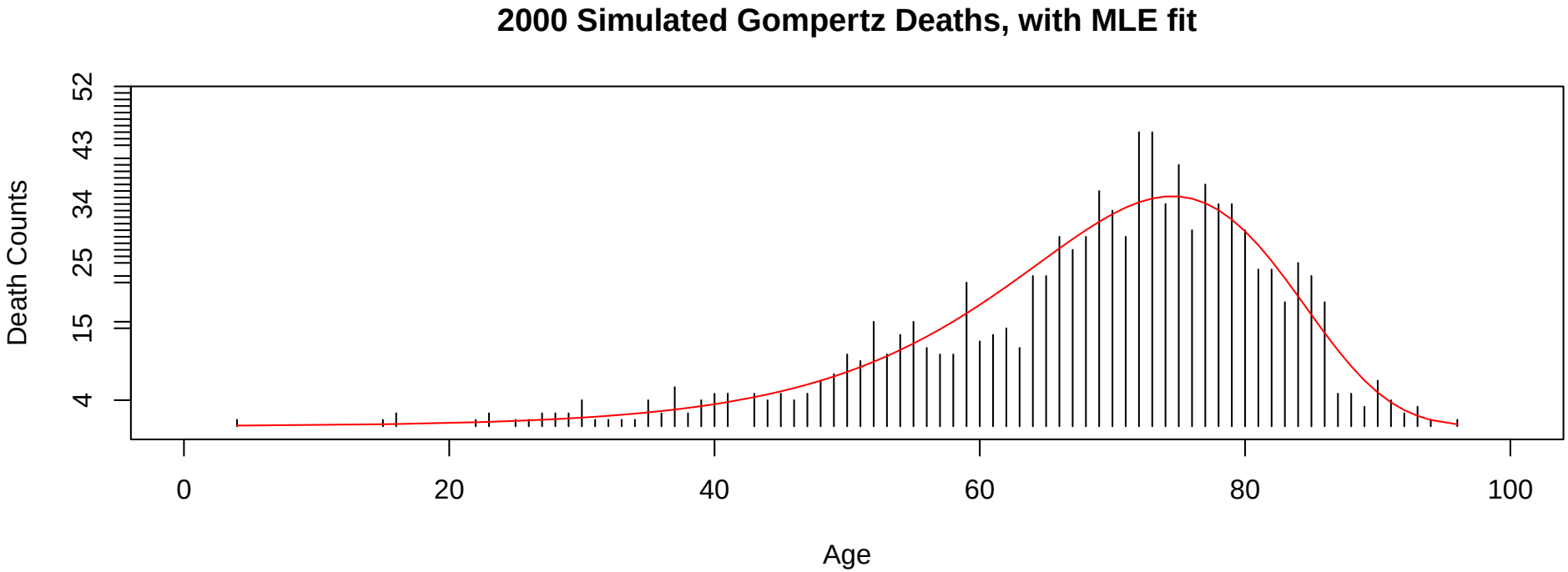
```
source("hmd_validation_functions.R")
N = 1000
set.seed(13)
x <- rgompertz.M(N, b = 1/10, M = 75)
Dx <- table(floor(x))
fit <- counts.trunc.gomp.est(Dx= Dx, x.left = 0, x.right = 200,
                             b.start = 1/9, M.start = 80)
(b.hat = exp(fit$par[1]))
```

```
## [1] 0.09572687
```

```
(M.hat = exp(fit$par[2]))
```

```
## [1] 75.01612
```

How did we do?


2000 Simulated Gompertz Deaths, with MLE fit
Death Counts
4
15
25
34
43
52
0
20
40
60
80
100
Age

Now artificially truncate to ages 65-90

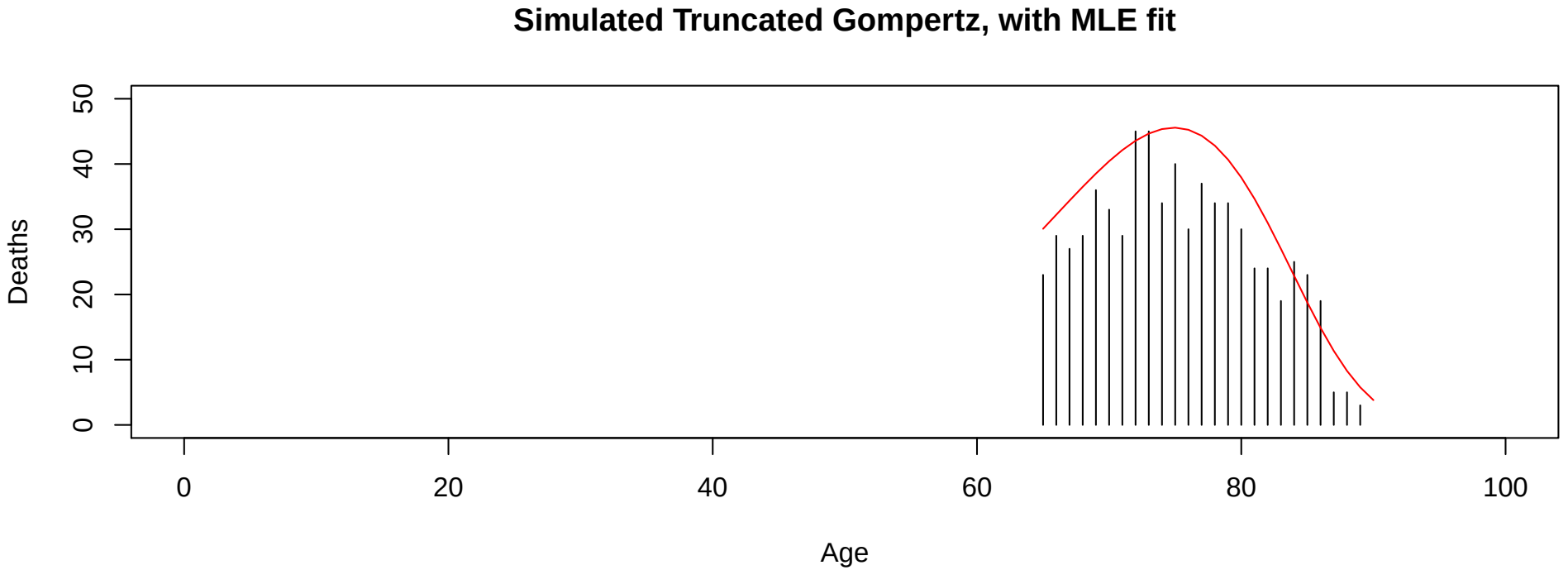
```
l <- 65
h <- 90
x.trunc <- x[x > l & x < h]
Dx <- table(floor(x.trunc))
fit <- counts.trunc.gomp.est(Dx= Dx, x.left = l, x.right = h,
                             b.start = 1/9, M.start = 80)
(b.hat = exp(fit$par[1]))
```

```
## [1] 0.1083535
```

```
(M.hat = exp(fit$par[2]))
```

```
## [1] 75.42279
```

Plot the fit


Simulated Truncated Gompertz, with MLE fit
Deaths
0
10
20
30
40
50
0
20
40
60
80
100
Age

Other ingredients

- A model
- Here we pick Gompertz, with parameters β and M.
- Some code to implement optimization routine
- Validatation
- Test on simulated Gompertz deaths, to see if we estimate right values
- Test on HMD to see if it works with real data
- Application
- NUMIDENT
- (Weighted Censoc)

Major Assumptions

- Gompertz model is appropriate
- Uniform coverage across ages to preserve the cohort distribution of deaths

Code

- Gompertz functions

```
source("hmd_validation_functions.R")

##
fit <- counts.trunc.gomp.est(Dx= Dx, x.left = 0, x.right = 200
                             b.start = 1/9, M.start = 80)
(b.hat = exp(fit$par[1]))
```

```
## [1] 0.1716812
```

```
(M.hat = exp(fit$par[2]))
```

```
## [1] 78.73153
```

- Optimization

```
## wrapper function that calls optim()
```

Without Truncation

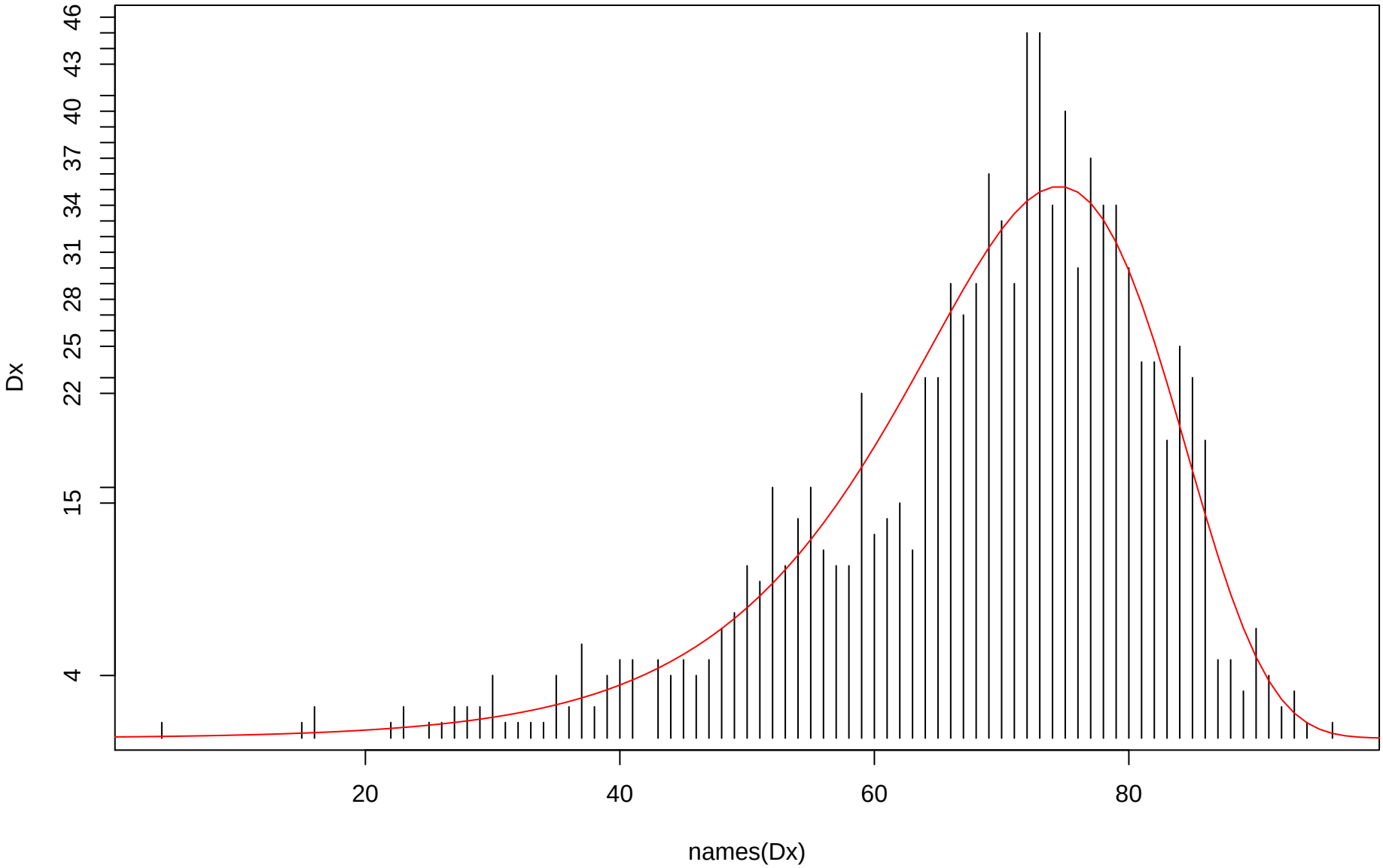
```
N = 1000
set.seed(13)
x <- rgompertz.M(N, b = 1/10, M = 75)
Dx <- table(floor(x))
fit <- counts.trunc.gomp.est(Dx= Dx, x.left = 0, x.right = 200,
                             b.start = 1/9, M.start = 80)
(b.hat = exp(fit$par[1]))
```

```
## [1] 0.09572687
```

```
(M.hat = exp(fit$par[2]))
```

```
## [1] 75.01612
```

Plot the fit



Validating Method with HMD

Our approach

- Use HMD death counts for the year and ages we have NUMIDENT deaths

(1988 to 2005, Ages 65 and over)

- Fit Gompertz and if we match
- Mortality rates
- e(65)

Import and prepare HMD data

```
library(data.table)
## read in and define age and cohort
dt.mx <- fread("~/Documents/hmd/hmd_statistics/death_rates/Mx_1x1/US
dt.mx[ , x := as.numeric(Age)]
```

```
## Warning in eval(jsub, SDenv, parent.frame()): NAs introduced by c
```

```
dt.mx[Age == "110+" , x := 110]
dt.mx[, cohort := Year - x]
dt <- fread("~/Documents/hmd/hmd_statistics/deaths/Deaths_1x1/USA.De
dt[ , x := as.numeric(Age)]
```

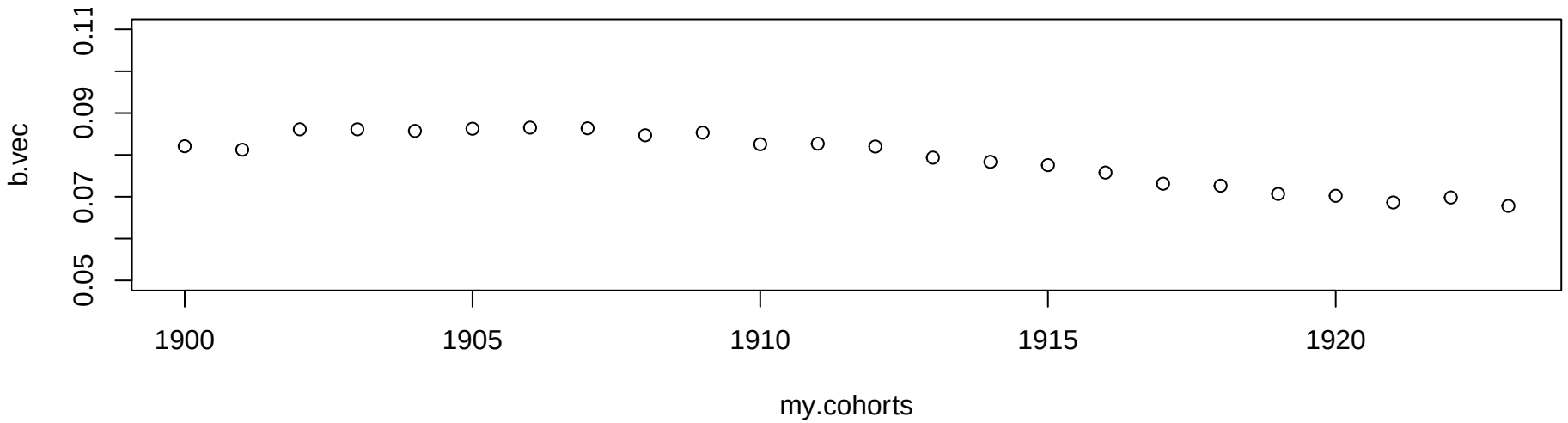
```
## Warning in eval(jsub, SDenv, parent.frame()): NAs introduced by c
```

```
dt[Age == "110+" , x := 110]
dt[, cohort := Year - x]
##
## make array, age x cohort x sex.
dt.long <- melt(dt, measure.vars = c("Male", "Female"), variable.na
```

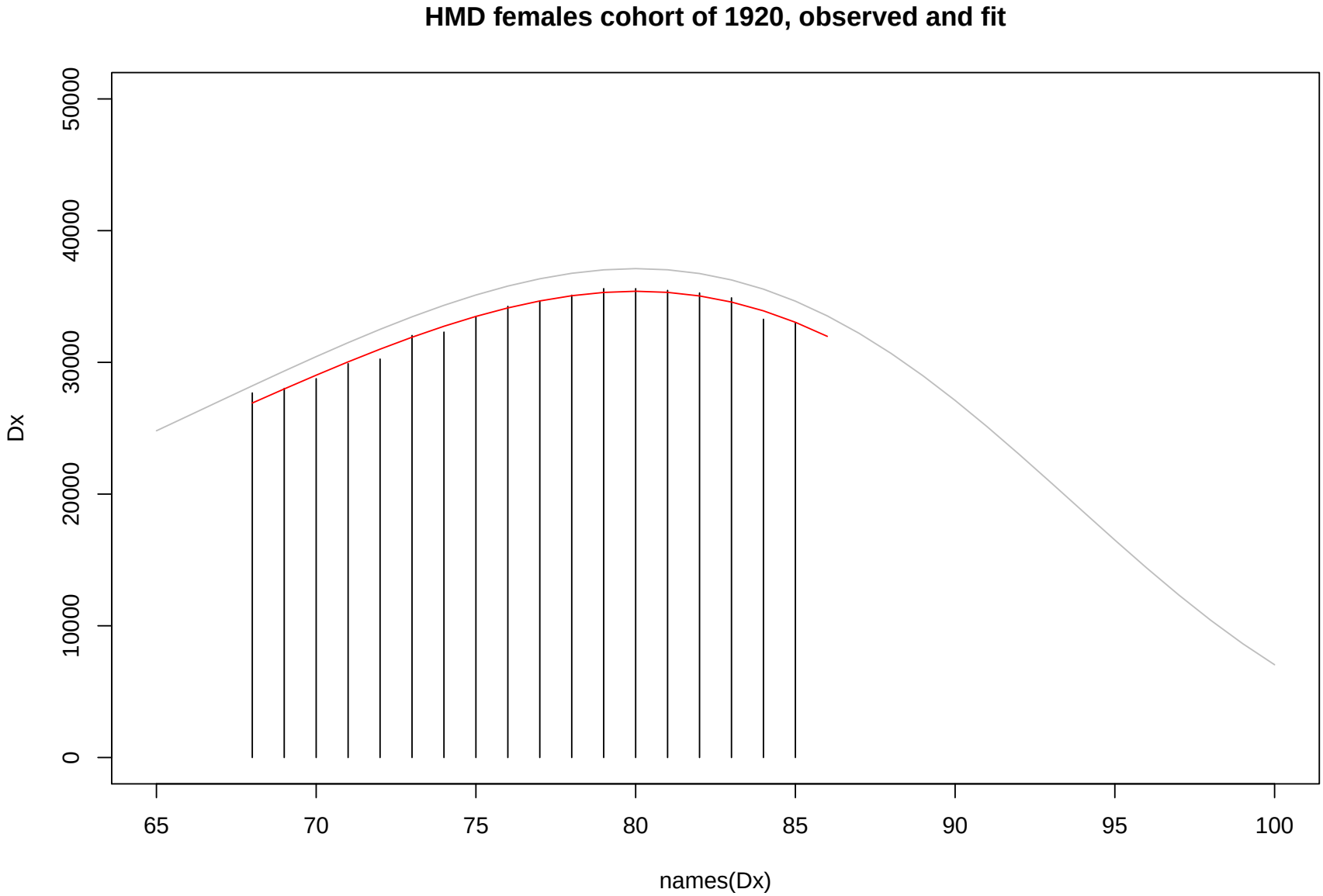
Fit HMD

```
## [1] 1900
## [1] 1901
## [1] 1902
## [1] 1903
## [1] 1904
## [1] 1905
## [1] 1906
## [1] 1907
## [1] 1908
## [1] 1909
## [1] 1910
## [1] 1911
## [1] 1912
## [1] 1913
## [1] 1914
## [1] 1915
## [1] 1916
```

Plot parameter values

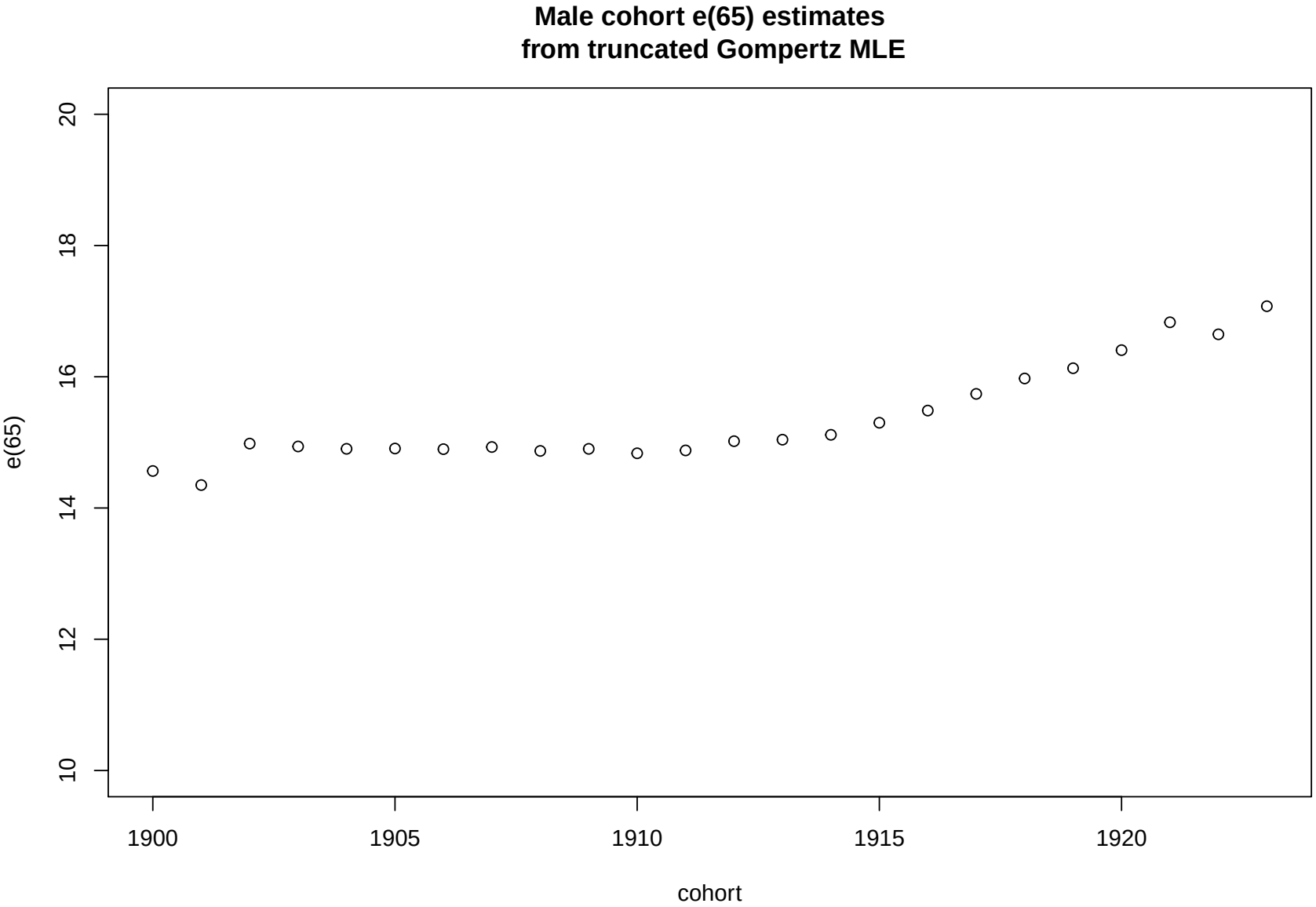


Look at cohort of 1920, pretending we observe only ages 68 to 85



Very high M is not crazy. Period life table for 2016 has mode at age 88

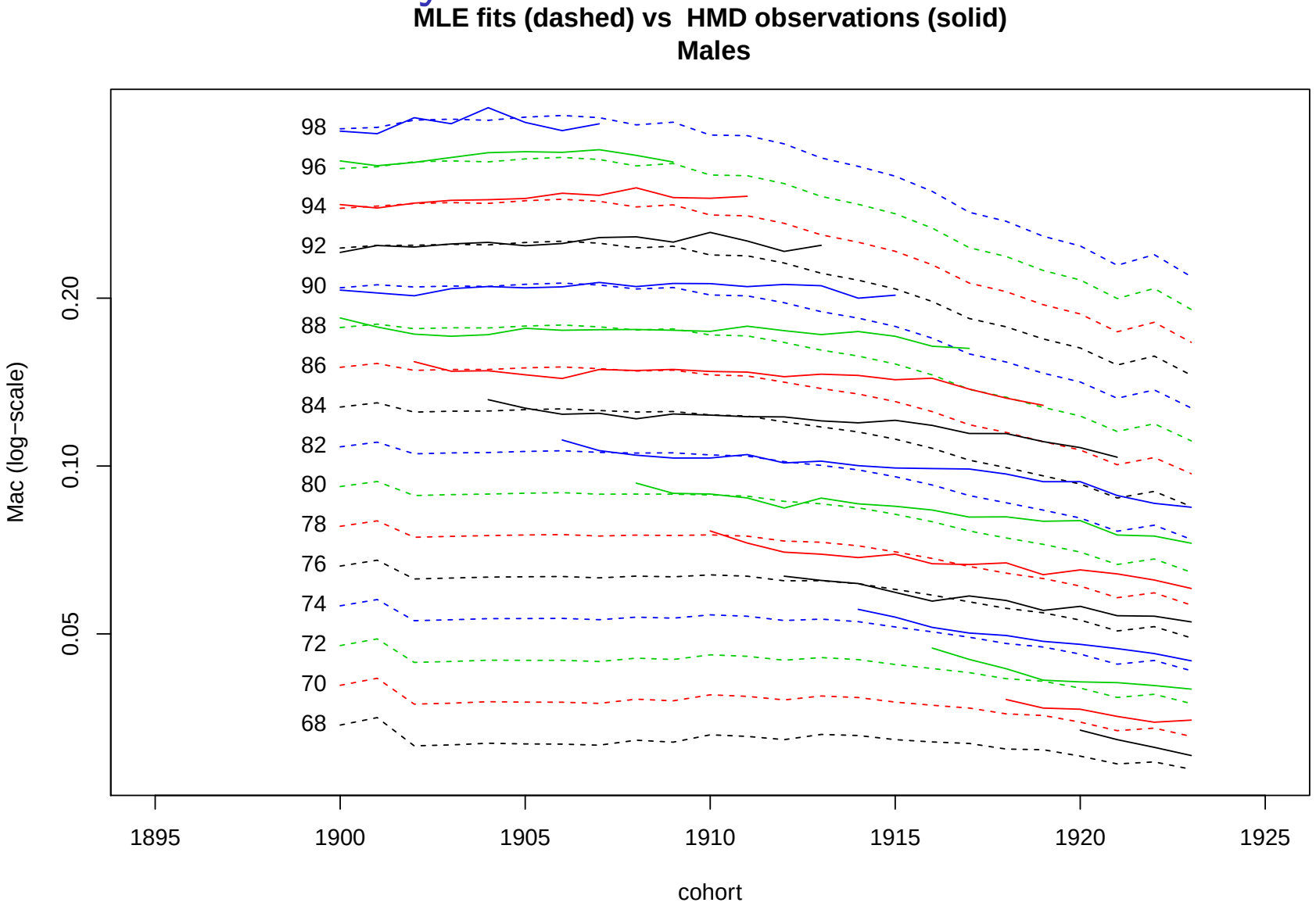
e(65)



For reference, period $e(65, 2016) = 18.36$.

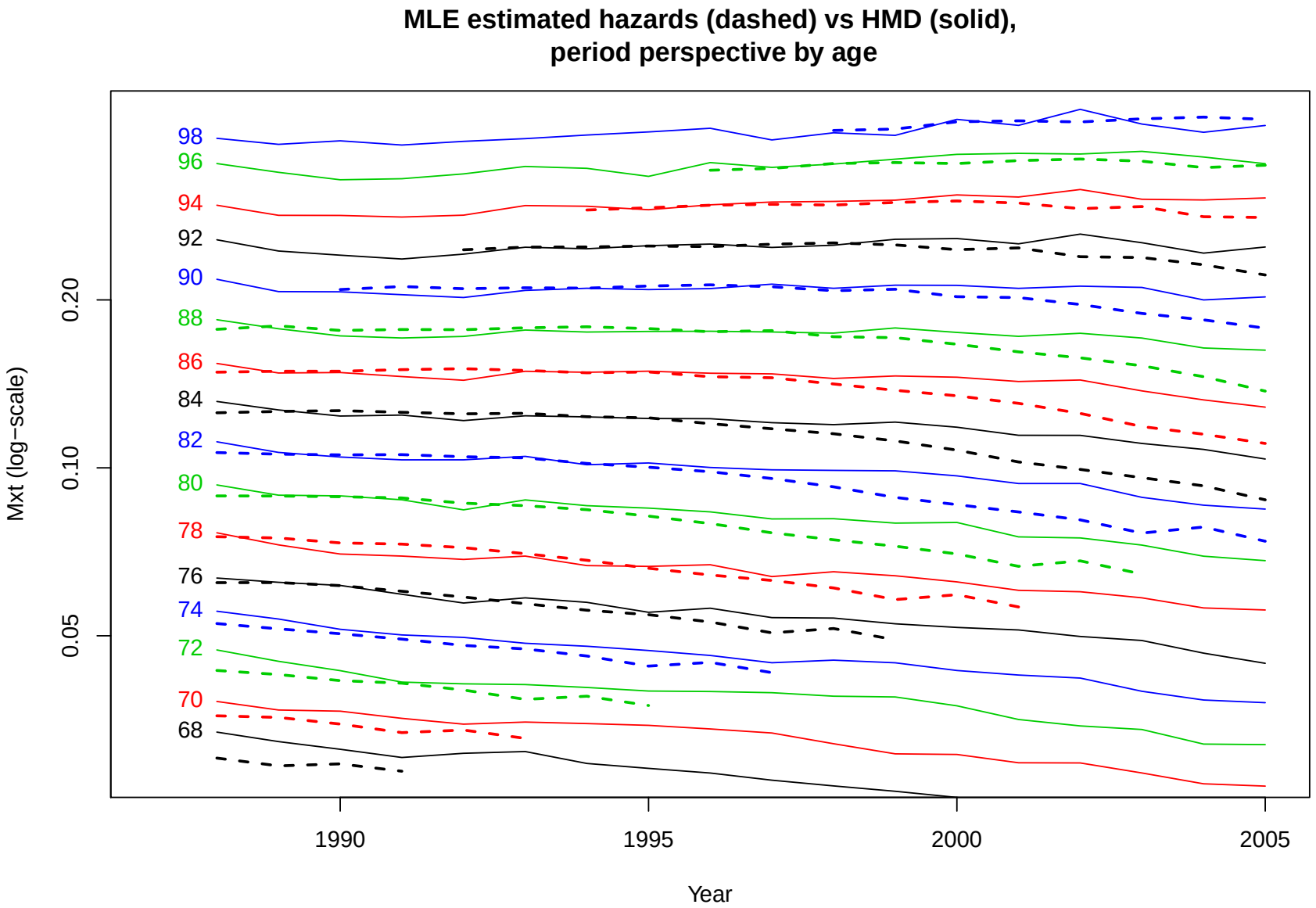
Mortality rates

Selected Mortality Rates



- Overall fit is remarkably good.
- But shouldn't rely on for unobserved ages (see old age decline in upper

Period perspective



Trying out the method with NUMIDENT